JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **37**, No. 3, August 2024
http://dx.doi.org/10.14403/jcms.2024.37.3.99

ADJOINT TRIPLES FOR FUZZY INFORMATION SYSTEMS

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Abstract. In this paper, we construct adjoint triples on lattices as non-commutative or non-associative algebraic structures. Moreover, we will construct adjoint triples on fuzzy sets which induce three types of fuzzy information systems which are the concept-forming operators, fuzzy relational erosions(dilations) and fuzzy relational property-oriented erosions(dilations). We give their examples.

1. Introduction

Complete residuated lattice, BL-algebra and quantale are important mathematical tools as algebraic structures for many valued logics [2,7-11,16-18]. However, these structures are very restrictive. As a weak condition with non-commutative or non-associative, Abdel-Hamid [1] introduced the notion of adjoint triples. Medina et al. [3-5] introduced the notion of algebraic structures [3-5], formal concept lattices [6,13,14], fuzzy relational mathematical morphology [12] and fuzzy logic programming [15] on an adjoint triple.

The main purpose of this paper is how to construct adjoint triples. In section 3, we construct adjoint triples with one of three operators as $\odot$, $\nearrow$ and $\searrow$. Moreover, we give their examples.

In section 4, let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Let X be a set of objects, Y be a set of attributes. We will construct three types of adjoint triples which induce three types of information systems from the following (1), (2) and (3).

Received August 25, 2024; Accepted September 12, 2024.

2020 Mathematics Subject Classification: Primary 03E72, 54A40; Secondary 54B10.

Key words and phrases: Adjoint triples, fuzzy information systems, the concept-forming operators, fuzzy relational (property-oriented) erosions(dilations).

* The present Research has been conducted by the Research Grant of Kwangwoon University in 2022.

(1) We can construct the adjoint triple $(\odot, \nearrow, \searrow)$ with respect to $(L_1^X, \leq_1)$, $(L_2^Y, \leq_2)$ and $((L_3^Y)^X, \leq_3)$. Form this result, for an information system $(X, Y, \psi \in (L_3^Y)^X)$, we can obtain the concept-forming operators (ref.[6,13,14]) $F : L_1^X \to L_2^Y$ and $G : L_2^Y \to L_1^X$ as follows

$$F(f)(y) = (f \nearrow \psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)),$$
$$G(g)(x) = (g \searrow \psi)(x) = \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)).$$

(2) We can construct the adjoint triple $(\odot, \nearrow, \searrow)$ with respect to $(L_1^X, \leq_1)$, $((L_2^Y)^X, \leq_2)$ and $(L_3^Y, \leq_3)$. Form this result, for an information system $(X, Y, \phi \in (L_2^Y)^X)$, we can obtain a fuzzy relational erosion and fuzzy relational dilation (ref.[6,13,14]) with respect to ϕ, $\epsilon_\phi : L_3^Y \to L_1^X$ and $\delta_\phi : L_1^X \to L_3^Y$ as follows

$$\epsilon_\phi(h)(x) = (\phi \searrow h)(x) = \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)),$$
$$\delta_\phi(f)(y) = (f \odot \phi)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y)).$$

(3) Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2$ $)$ and $(L_3, \leq_3)$. Let X be a set of objects, Y be a set of attributes and the triple $(X, Y, \theta \in (L_1^Y)^X)$ be an information system. From the adjoint triple $(\odot, \nearrow, \searrow)$ with respect to $((L_1^Y)^X, \leq_1)$, $(L_2^Y, \leq_2)$ and $(L_3^X, \leq_3)$ in above corollary, we construct a fuzzy relational property-oriented erosion and fuzzy relational property-oriented dilation (ref.[6,13,14]) with respect to θ, $\epsilon_\theta : L_3^X \to L_2^Y$ and $\delta_\phi : L_2^Y \to L_3^X$ is defined as

$$\epsilon_\theta(h)(y) = (\theta \nearrow h)(y) = \bigwedge_{x\in X}(\theta(x)(y) \nearrow h(x)),$$
$$\delta_\theta(f)(x) = (\theta \odot g)(x) = \bigvee_{y\in Y}(\theta(x)(y) \odot g(y)).$$

2. Preliminaries

Definition 2.1. [3-5] Let $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$ be complete lattices. We say that for the mappings $\odot : L_1 \times L_2 \to L_3$, $\searrow : L_2 \times L_3 \to L_1$ and $\nearrow : L_1 \times L_3 \to L_2$, $(\odot, \nearrow, \searrow)$ is called an *adjoint triple* with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$ if it satisfies the following conditions:

$x \leq_1 y \searrow z$ iff $x \odot y \leq_3 z$ iff $y \leq_2 x \nearrow z$ for $x \in L_1, y \in L_2, z \in L_3$.

Lemma 2.2. [3-5] *Let L_i be lattices for $i = 1, 2, 3$. Let $(\odot, \searrow, \nearrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$. For each $x, x_1, x_2 \in L_1, y, y_1, y_2 \in L_2, z, z_1, z_2 \in L_3$, we have the following properties.*

(1) *If $x_1 \leq_1 x_2$, then $x_1 \odot y \leq_3 x_2 \odot y$ and $x_2 \nearrow z \leq_2 x_1 \nearrow z$.*

(2) *If $y_1 \leq_2 y_2$, then $x \odot y_1 \leq x \odot y_2$ and $y_2 \searrow z \leq_1 y_1 \searrow z$.*

(3) *If* $z_1 \leq_3 z_2$, *then* $x \nearrow z_1 \leq_2 x \nearrow z_2$ *and* $y \searrow z_1 \leq_1 y \searrow z_2$.
(4) $y \leq_2 x \nearrow (x \odot y)$, $x \leq_1 y \searrow (x \odot y)$.
(5) $x \odot (x \nearrow z) \leq_3 z$, $(y \searrow z) \odot y \leq_3 z$.
(6) $y \leq_2 (y \searrow z) \nearrow z$, $x \leq_1 (x \nearrow z) \searrow z$.

3. Construction of adjoint triples

In this section, we construct an adjoint triple with one of three operators as $\odot$, $\nearrow$ and $\searrow$.

THEOREM 3.1. *Let* $(L_1, \leq_1)$, $(L_2, \leq_2)$ *and* $(L_3, \leq_3)$ *be lattices and* $\odot : L_1 \times L_2 \to L_3$ *be a map. If* $x \odot (\bigvee_{j\in J} y_i) = \bigvee_{j\in J}(x \odot y_j)$ *for each* $\{y_j\}_{j\in J}$ *and* $\bigvee\{y \in L_2 \mid x \odot y \leq_3 z\}$ *exists for each* $x \in L_1, z \in L_3$. *then the following statements* (1), (2) *and* (3) *are equivalent.*

(1) *If* $z_1 \leq_3 z_2$, *then* $x \nearrow z_1 \leq_2 x \nearrow z_2$. *Moreover,* $x \odot (x \nearrow z) \leq_3 z$ *and* $y \leq_2 x \nearrow (x \odot y)$.

(2) $x \nearrow z = \bigvee\{y \in L_2 \mid x \odot y \leq_3 z\}$.

(3) $x \odot y \leq_3 z$ *iff* $y \leq_2 x \nearrow z$.

Proof. (1) $\Rightarrow$ (2). Put $P(x, z) = \bigvee\{y \in L \mid x \odot y \leq_3 z\}$. By (1), since $x \odot (x \nearrow z) \leq_3 z$, $P(z, x) \geq_2 x \nearrow z$.

Suppose there exist $x \in L_1, z \in L_3$ such that $P(x, z) \not\leq_2 x \nearrow z$. Then there exists $y \in L_2$ such that $y \not\leq_2 x \nearrow z$ and $x \odot y \leq_3 z$. By (1),

$$y \leq_2 x \nearrow (x \odot y) \leq_2 x \nearrow z.$$

It is a contradiction. Hence $P(x, z) \leq_2 x \nearrow z$. Thus, $x \nearrow z = P(x, z) = \bigvee\{y \in L \mid x \odot y \leq_3 z\}$.

(2) $\Rightarrow$ (3). Let $x \odot y \leq_3 z$. Then $y \leq_2 x \nearrow z$.

If $y \leq_2 x \nearrow z$, then $x \odot y \leq_3 x \odot (x \nearrow z) = x \odot \bigvee\{y_1 \in L_2 \mid x \odot y_1 \leq_3 z\}) = \bigvee(x \odot y_1) \leq_3 z$.

(3) $\Rightarrow$ (1). Since $x \odot y \leq_3 x \odot y$, $y \leq_2 x \nearrow (x \odot y)$. Since $x \nearrow z \leq_2 x \nearrow z$, $x \odot (x \nearrow z) \leq_3 z$. If $z_1 \leq_3 z_2$, $x \odot (x \nearrow z_1) \leq_3 z_1 \leq_3 z_2$. Hence $x \nearrow z_1 \leq_2 x \nearrow z_2$.

□

THEOREM 3.2. *Let* $(L_1, \leq_1)$, $(L_2, \leq_2)$ *and* $(L_3, \leq_3)$ *be lattices and* $\odot : L_1 \times L_2 \to L_3$ *be a map. If* $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in I}(x_i \odot y)$ *for each* $\{x_i\}_{i\in I}$ *and* $\bigvee\{x \in L_1 \mid x \odot y \leq_3 z\}$ *exists for each* $y \in L_2, z \in L_3$, *then* (1), (2) *and* (3) *are equivalent.*

(1) *If* $z_1 \leq_3 z_2$, *then* $y \searrow z_1 \leq_1 y \searrow z_2$. *Moreover, for all* $x, y \in L$, $(y \searrow z) \odot y \leq_3 z$ *and* $x \leq_1 y \searrow (x \odot y)$.

(2) $y \searrow z = \bigvee\{x \in L_1 \mid x \odot y \leq_3 z\}$.

(3) $x \odot y \leq_3 z$ *iff* $x \leq_1 y \searrow z$.

Proof. (1) $\Rightarrow$ (2). Put $Q(y,z) = \bigvee\{x \in L_1 \mid x \odot y \leq_3 z\}$. By (1), since $(y \searrow z) \odot y \leq_3 z$, $Q(y,z) \geq_1 y \searrow z$.

Suppose there exist $z \in L_3, y \in L_2$ such that $Q(y,z) \not\leq_1 y \searrow z$. Then there exists $x \in L_1$ such that $x \not\leq_1 y \searrow z$ and $x \odot y \leq_3 z$. By (1),

$$x \leq_1 y \searrow (x \odot y) \leq_1 y \searrow z.$$

It is a contradiction. Hence $Q(y,z) \leq_1 y \searrow z$. Thus, $y \searrow z = Q(y,z) = \bigvee\{x \in L_1 \mid x \odot y \leq_3 z\}$.

(2) $\Rightarrow$ (3). Let $x \odot y \leq_3 z$. Then $x \leq_1 y \searrow z$.

If $x \leq_1 y \searrow z$, then $x \odot y \leq_3 (y \searrow z) \odot y = \bigvee\{x_2 \in L_1 \mid x_2 \odot y \leq_3 z\}) = \bigvee(x_2 \odot y) \leq_3 z$.

(3) $\Rightarrow$ (1). Since $x \odot y \leq_3 x \odot y$, $x \leq_1 y \searrow (x \odot y)$. Since $y \searrow z \leq_1 y \searrow z$, $(y \searrow z) \odot y \leq_3 z$. If $z_1 \leq_3 z_2$, $(y \searrow z_1) \odot y \leq_3 z_1 \leq_3 z_2$. Hence $y \searrow z_1 \leq_1 y \searrow z_2$.

□

From Theorems 3.1 and 3.2, we can obtain the following corollary.

Corollary 3.3. *Let $(L_i, \vee, \wedge, \top, \bot)$ be a bounded lattice for $i \in \{1,2,3\}$ and $\odot : L_1 \times L_2 \to L_3$ be a map. Let $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in I}(x_i \odot y)$ for each $\{x_i\}_{i\in I}$ and $\bigvee\{x \in L_1 \mid x \odot y \leq_3 z\}$ exists for each $y \in L_2, z \in L_3$ and we define*

$$y \searrow z = \bigvee\{x \in L_1 \mid x \odot y \leq_3 z\}.$$

Moreover, let $x \odot (\bigvee_{j\in J} y_i) = \bigvee_{j\in J}(x \odot y_j)$ for each $\{y_j\}_{j\in J}$ and $\bigvee\{y \in L_2 \mid x \odot y \leq_3 z\}$ exists for each $x \in L_1, z \in L_3$ and we define $\nearrow$: $L_1 \times L_3 \to L_2$ as

$$x \nearrow z = \bigvee\{y \in L_2 \mid x \odot y \leq_3 z\}.$$

Then $(\odot, \nearrow, \searrow)$ is an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$.

Example 3.4. (1) Let $(\{x_0,x_1,x_2\}, x_0 <_1 x_1 <_1 x_2)$, $(\{y_0,y_1,y_2,y_3\}, y_0 <_2 y_1 <_2 y_2 <_2 y_3)$ and $(\{z_0,z_1,z_2\}, z_0 <_3 z_1 <_3 z_2)$ be lattices. Define $\odot_1, \odot_2 : L_1 \times L_2 \to L_3$ as follows:

$\odot_1$	y_0	y_1	y_2	y_3
x_0	z_0	z_0	z_0	z_0
x_1	z_0	z_1	z_1	z_2
x_2	z_0	z_1	z_2	z_2

$\odot_2$	y_0	y_1	y_2	y_3
x_0	z_0	z_0	z_1	z_2
x_1	z_0	z_1	z_1	z_2
x_2	z_0	z_1	z_2	z_2

Then, for each $k = 1, 2$, $(\bigvee_{i\in I} x_i) \odot_k y = \bigvee_{i\in I}(x_i \odot_k y)$ for each $y \in L_2, I \subset \{0,1,2\}$ and $x \odot_k (\bigvee_{j\in J} y_j) = \bigvee_{j\in J}(x \odot_k y_j)$ for each $x \in L_1, J \subset \{0,1,2,3\}$.

Since $\bigvee\{x \in L_1 \mid x \odot_1 y \leq_3 z\}$ exists for each $y \in L_2, z \in L_3$ and we define $z \searrow_1 y = \bigvee\{x \in L_1 \mid x \odot_1 y \leq_3 z\}$. Moreover, $\bigvee\{y \in L_2 \mid x \odot_1 y \leq_3 z\}$ exists for each $x \in L_1, z \in L_3$, we can define $x \nearrow z = \bigvee\{y \in L_2 \mid x \odot y \leq_3 z\}$. Then $(\odot_1, \nearrow_1, \searrow_1)$ is an adjoint triple with

$\searrow_1$	z_0	z_1	z_2
y_0	x_2	x_2	x_2
y_1	x_0	x_2	x_2
y_2	x_0	x_1	x_2
y_3	x_0	x_0	x_2

$\nearrow_1$	z_0	z_1	z_2
x_0	y_3	y_3	y_3
x_1	y_0	y_2	y_3
x_2	y_0	y_1	y_3

Since $\{x \in L_1 \mid x \odot_2 y_2 \leq_3 z_0\} = \emptyset$, there does not exist $y_2 \searrow_2 z_0$. Moreover, there do neither exist $y_3 \searrow_2 z_0$ nor $y_3 \searrow_2 z_1$.

$\searrow_2$	z_0	z_1	z_2
y_0	x_1	x_1	x_2
y_1	x_0	x_1	x_2
y_2	-	x_1	x_2
y_3	-	-	x_2

$\nearrow_2$	z_0	z_1	z_2
x_0	y_3	y_3	y_3
x_1	y_0	y_2	y_3
x_2	y_0	y_1	y_3

Hence $(\odot_2, \nearrow_2, \searrow_2)$ is not an adjoint triple.

(2) A complete lattice $(L, \leq, \odot)$ is called a quantale [11] if it satisfies

(CQ1) $a \odot (b \odot c) = (a \odot b) \odot c$ for all $a, b, c \in L$;

(CQ2) $(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma}(a_i \odot b)$ and $a \odot (\bigvee_{i\in\Gamma} b_i) = \bigwedge_{i\in\Gamma}(a \odot b_i)$ for all $a, a_i, b, b_i \in L$.

Define $a \searrow b = \bigvee\{x \in L \mid x \odot a \leq b\}$ and $a \nearrow b = \bigvee\{y \in L \mid a \odot y \leq b\}$. Then $(\odot, \nearrow, \searrow)$ is an adjoint triple with respect to $(L_i = L, \leq_i = \leq)$, for $i = 1, 2, 3$.

(3) Define $\odot : [0,1] \times [0,1] \to [0,1]$ by

$$x \odot y = \begin{cases} 0, & \text{if } 0 \leq x \leq 0.3, 0 \leq y \leq 0.4, \\ x \wedge y, & \text{otherwise.} \end{cases}$$

Then $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$ and $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ for all $x, x_i, y, y_i \in [0,1]$. But

$$0 = 0.2 \odot (\bigwedge_{n\in N}(0.4 + \tfrac{1}{n})) \neq \bigwedge_{n\in N}(0.2 \odot (0.4 + \tfrac{1}{n})) = 0.2,$$
$$0 = (\bigwedge_{n\in N}(0.3 + \tfrac{1}{n})) \odot 0.1 \neq \bigwedge_{n\in N}((0.3 + \tfrac{1}{n}) \odot 0.1) = 0.1.$$

We can obtain $\searrow, \nearrow$: $[0,1]\times[0,1]\to[0,1]$ as $x \nearrow y = \bigvee\{z\in[0,1] \mid x\odot z \le y\}$, $x \searrow y = \bigvee\{z\in[0,1] \mid z\odot x\le y\}$. Then $(\odot, \searrow, \nearrow)$ is an adjoint triple with

$$x \nearrow y = \begin{cases} 0.4, & \text{if } x\le 0.3, x>y \\ y, & \text{if } x>0.3, x>y \\ 1, & \text{if } x\le y, \end{cases}$$

$$x \searrow y = \begin{cases} 0.3\vee y, & \text{if } x\le 0.4, x>y \\ y, & \text{if } x>0.4, x>y \\ 1, & \text{if } x\le y. \end{cases}$$

THEOREM 3.5. *Let $(L_1,\le_1)$, $(L_2,\le_2)$ and $(L_3,\le_3)$ be lattices and $\nearrow$: $L_1\times L_3\to L_2$ be a map.*

(1) *If $x\nearrow(\bigwedge_{j\in J} z_i) = \bigwedge_{j\in J}(x\nearrow z_j)$ for each $\{z_j\}_{j\in J}$ and $\bigwedge\{z\in L_3 \mid x\nearrow z\ge_2 y\}$ exists for each $x\in L_1, y\in L_2$ and define $\odot: L_1\times L_2\to L_3$ as*

$$x\odot y = \bigwedge\{z\in L_3 \mid x\nearrow z\ge_2 y\},$$

then $x\odot y\le_3 z$ iff $y\le_2 x\nearrow z$.

(2) *If $(\bigvee_{i\in I} x_i)\nearrow z = \bigwedge_{i\in I}(x_i\nearrow z)$ for each $\{x_i\}_{i\in I}$ and $\bigvee\{x\in L_1 \mid x\nearrow z\ge_2 y\}$ exists for each $z\in L_3, y\in L_2$ and define $\searrow$: $L_2\times L_3\to L_1$ as*

$$y\searrow z = \bigvee\{x\in L_1 \mid x\nearrow z\ge_2 y\},$$

then $x\le_1 y\searrow z$ iff $y\le_2 x\nearrow z$.

Proof. (1) Define $f_x: L_3\to L_2$ as $f_x(z) = x\nearrow z$. Then $f_x(\bigwedge_{j\in J} z_i) = x\nearrow(\bigwedge_{j\in J} z_i) = \bigwedge_{j\in J}(x\nearrow z_j) = \bigwedge_{j\in J} f_x(z_i)$. There exists $K_{f_x}: L_2\to L_3$ such that

$$K_{f_x}(y) = \bigwedge\{z\in L_3 \mid x\nearrow z\ge_1 y\}.$$

Define $x\odot y = K_{f_x}(y)$. Then if $x\nearrow z\ge_1 y$, then $K_{f_x}(y)\le_3 z$.

If $K_{f_x}(y)\le_3 z$, then $x\nearrow z\ge_1 x\nearrow K_{f_x}(y)\ge_1 y$. Hence $K_{f_x}(y) = x\odot y\le_3 z$ iff $y\le_2 x\nearrow z = f_x(z)$.

(2) Define $f^z: L_1\to L_2$ as $f^z(x) = x\nearrow z$. Then $f^z(\bigvee_{i\in I} x_i) = (\bigvee_{i\in I} x_i)\nearrow z = \bigwedge_{i\in I}(x_i\nearrow z) = \bigwedge_{i\in I} f^z(x_i)$. There exists $K_{f^z}: L_2\to L_1$ such that

$$K_{f^z}(y) = \bigvee\{x\in L_1 \mid x\nearrow z\ge_2 y\}.$$

Define $K_{f^z}(y) = y\searrow z$.

If $x\nearrow z\ge_2 y$, then $K_{f^z}(y)\ge_2 x$.

If $K_{f^z}(y)\ge_3 z$, then $K_{f^z}(y)\nearrow z = \bigwedge_{w\in\{x\in L_1 \mid x\nearrow z\ge_2 y\}}(w\nearrow z)\ge_2 y$.

Hence $x\le_1 K_{f^z}(y) = y\searrow z$ iff $y\le_2 f^z(x) = x\nearrow z$.

□

Corollary 3.6. *Let $(L_1,\leq_1)$, $(L_2,\leq_2)$ and $(L_3,\leq_3)$ be lattices and $\searrow: L_2\times L_3\to L_1$ be a map.*

(1) *If $y\searrow(\bigwedge_{j\in J} z_i)=\bigwedge_{j\in J}(y\searrow z_j)$ for each $\{z_j\}_{j\in J}$ and $\bigwedge\{z\in L_3\mid y\searrow z\geq_1 x\}$ exists for each $x\in L_1, y\in L_2$ and define $\odot: L_1\times L_2\to L_3$ as*

$$x\odot y=\bigwedge\{z\in L_3\mid y\searrow z\geq_1 x\},$$

then $x\odot y\leq_3 z$ iff $x\leq_1 y\searrow z$.

(2) *If $(\bigvee_{i\in I} y_i)\searrow z=\bigwedge_{i\in I}(y_i\searrow z)$ for each $\{y_i\}_{i\in I}$ and $\bigvee\{y\in L_2\mid y\searrow z\geq_1 x\}$ exists for each $z\in L_3, x\in L_2$ and define $\nearrow: L_1\times L_3\to L_2$ as*

$$x\nearrow z=\bigvee\{y\in L_2\mid y\nearrow z\geq_1 x\},$$

then $y\leq_2 x\nearrow z$ iff $x\leq_1 y\searrow z$.

Example 3.7. (1) Let $(L_1,\leq_1)=(\{x_0,x_1,x_2\},x_0<_1 x_1<_1 x_2)$, $(L_2,\leq_2)=(\{y_0,y_1,y_2,y_3\},y_0<_2 y_1<_2 y_2<_2 y_3)$ and $(L_3,\leq_3)=(\{z_0,z_1,z_2\},z_0<_3 z_1<_3 z_2)$ be lattices. Define $\nearrow_1,\nearrow_2: L_1\times L_3\to L_2$ as follows:

$\nearrow_1$	z_0	z_1	z_2
x_0	y_3	y_3	y_3
x_1	y_1	y_2	y_3
x_2	y_0	y_1	y_3

$\nearrow_2$	z_0	z_1	z_2
x_0	y_3	y_3	y_3
x_1	y_0	y_2	y_3
x_2	y_0	y_1	y_2

Since $x\nearrow_1(\bigwedge_{j\in J} z_i)=\bigwedge_{j\in J}(x\nearrow_1 z_j)$ for each $\{z_j\}_{j\in J}$ and $\bigwedge\{z\in L_3\mid x\nearrow_1 z\geq_2 y\}$ exists for each $x\in L_1, y\in L_2$, we can obtain $\odot_1: L_1\times L_2\to L_3$ as

$$x\odot_1 y=\bigwedge\{z\in L_3\mid x\nearrow_1 z\geq_2 y\}.$$

Since $\{z\in L_3\mid x_2\nearrow_2 z\geq_2 y_3\}=\emptyset$, $x_2\odot_2 y_3$ dose not exist from the following table.

$\odot_1$	y_0	y_1	y_2	y_3
x_0	z_0	z_0	z_0	z_0
x_1	z_0	z_0	z_1	z_2
x_2	z_0	z_1	z_2	z_2

$\odot_2$	y_0	y_1	y_2	y_3
x_0	z_0	z_0	z_0	z_0
x_1	z_0	z_1	z_1	z_2
x_2	z_0	z_1	z_2	$-$

Since $\nearrow_1$ and $\nearrow_2$ satisfy the conditions of Theorem 3.5(2), $\searrow_1$ and $\searrow_2$ exist as follows.

$\searrow_1$	z_0	z_1	z_2
y_0	x_2	x_2	x_2
y_1	x_1	x_2	x_2
y_2	x_1	x_1	x_2
y_3	x_0	x_0	x_2

$\searrow_2$	z_0	z_1	z_2
y_0	x_2	x_2	x_2
y_1	x_0	x_2	x_2
y_2	x_0	x_1	x_2
y_3	x_0	x_0	x_1

Hence $(\odot_1, \nearrow_1, \searrow_1)$ is an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$. But $(\odot_2, \nearrow_2, \searrow_2)$ is not an adjoint triple.

4. Adjoint triples for fuzzy information systems

In this section, we construct adjoint triples for fuzzy information systems.

LEMMA 4.1. *Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1$), $(L_2, \leq_2)$ and $(L_3, \leq_3)$. Then the following properties hold.*

(1) $(\bigvee_{i\in I} x_i) \odot y = \bigvee_{i\in I}(x_i \odot y)$ for each $\{x_i\}_{i\in I}$ and $x \odot (\bigvee_{j\in J} y_i) = \bigvee_{j\in J}(x \odot y_j)$ for each $\{y_j\}_{j\in J}$ where the arbitrary joins exist.

(2) $x \nearrow (\bigwedge_{i\in I} z_i) = \bigwedge_{i\in I}(x \nearrow z_i)$ and $(\bigvee_{i\in I} x_i) \nearrow z = \bigwedge_{i\in I}(x_i \nearrow z)$ where the arbitrary join and meets exist.

(3) $y \searrow (\bigwedge_{i\in I} z_i) = \bigwedge_{i\in I}(y \searrow z_i)$ and $(\bigvee_{i\in I} y_i) \searrow z = \bigwedge_{i\in I}(y_i \searrow z)$ where the arbitrary join and meets exist.

Proof. (1) Let $x_1 \leq_1 x_2$. Then $x_1 \leq_1 x_2 \leq_1 y_2 \searrow x_2 \odot y_2$. Hence $x_1 \odot y_2 \leq_3 x_2 \odot y_2$. Thus, $\bigvee_{i\in I}(x_i \odot y) \leq_3 (\bigvee_{i\in \Gamma} x_i) \odot y$.

Since $x_i \leq_1 y \searrow \bigvee_{i\in I}(x_i \odot y)$, $\bigvee_{i\in I} x_i \leq_1 y \searrow \bigvee_{i\in I}(x_i \odot y)$. Hence $(\bigvee_{i\in \Gamma} x_i) \odot y \leq_3 \bigvee_{i\in I}(x_i \odot y)$. Thus $\bigvee_{i\in I}(x_i \odot y) = (\bigvee_{i\in \Gamma} x_i) \odot y$.

(2) Let $z_1 \leq_3 z_2$. Since $x \odot (x \nearrow z_1) \leq_3 z_1 \leq_3 z_2$, $x \nearrow z_1 \leq_2 x \nearrow z_2$. Hence $x \nearrow (\bigwedge_i z_i) \leq_2 \bigwedge_i (x \searrow z_i)$.

Since $x \odot \bigwedge_i (x \searrow z_i) \leq_3 \bigwedge_i (x \odot (x \searrow z_i)) \leq_3 \bigwedge_i z_i$, $\bigwedge_i (x \searrow z_i) \leq_2 x \nearrow (\bigwedge_i z_i)$. Hence $\bigwedge_i (x \searrow z_i) = x \nearrow (\bigwedge_i z_i)$.

Let $x_1 \leq_1 x_2$. Since $x_1 \odot (x_2 \nearrow z) \leq_3 x_2 \odot (x_2 \nearrow z) \leq_3 z$, $x_2 \nearrow z \leq_2 x_1 \nearrow z$. Thus, $(\bigvee_i x_i) \nearrow z \leq_2 \bigwedge_i (x_i \nearrow z)$. Moreover, $p \leq_2 x_i \nearrow z$ for all $i \in I$, $x_i \leq_1 p \searrow z$. Then $\bigvee_{i\in I} x_i \leq_1 p \searrow z$. Hence $p \leq_2 \bigvee_{i\in I} x_i \nearrow z$. So, $(\bigvee_i x_i) \nearrow z = \bigwedge_i (x_i \nearrow z)$.

(3) It is similarly proved as (2).

□

THEOREM 4.2. *Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$ and X, Y be sets. Define $\odot : L_1^X \times$*

$L_2^Y \to (L_3^Y)^X$ *as* $(f \odot g)(x)(y) = f(x) \odot g(y)$. *Then there exist* $\nearrow$: $L_1^X \times (L_3^Y)^X \to L_2^Y$ *and* $\searrow$: $L_2^Y \times (L_3^Y)^X \to L_1^X$ *defined as*

$$(f \nearrow \psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)),$$
$$(g \searrow \psi)(x) = \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)).$$

Moreover, $(\odot, \nearrow, \searrow)$ *is an adjoint triple with respect to* $(L_1^X, \leq_1)$, $(L_2^Y, \leq_2)$ *and* $((L_3^Y)^X, \leq_3)$ *such that*

$$f \leq_1 g \searrow \psi \text{ iff } f \odot g \leq_3 \psi \text{ iff } g \leq_2 f \nearrow \psi.$$

Proof. Put $\Pi_f(g)(x)(y) = f(x) \odot g(y)$ for a fixed $f \in L_1^X$. By Lemma 4.1 (1), $\Pi_f(\bigvee_{i\in\Gamma} g_i) = f \odot (\bigvee_{i\in\Gamma} g_i) = \bigvee_{i\in\Gamma}(f \odot g_i) = \bigvee_{i\in\Gamma} \Pi_f(g_i)$. Define $\mathcal{H}_{\Pi_f} : (L_3^Y)^X \to L_2^Y$ as

$$\mathcal{H}_{\Pi_f}(\psi) = \bigvee\{g \in L_2^Y \mid \Pi_f(g)(x)(y) \leq_3 \psi(x)(y)\}.$$

Since $\Pi_f(g)(x)(y) = f(x) \odot g(y) \leq_3 \psi(x)(y)$, $g(y) \leq_2 \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y))$. Hence $\mathcal{H}_{\Pi_f}(\psi)(y) \leq_2 \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y))$.

On the other hand, since $\Pi_f(f \nearrow \psi)(x)(y) = f(x) \odot \bigwedge_{z\in X}(f(z) \nearrow \psi(z)(y)) \leq_3 f(x) \odot (f(x) \nearrow \psi(x)(y)) \leq_3 \psi(x)(y)$, we have $\mathcal{H}_{\Pi_f}(\psi)(y) \geq_2 \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y))$. Thus, $\mathcal{H}_{\Pi_f}(\psi)(y) = (f \nearrow \psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y))$. It follows

$$\Pi_f(g)(x)(y) = f(x) \odot g(y) \leq_3 \psi(x)(y)$$
$$\text{iff } g(y) \leq_2 \mathcal{H}_{\Pi_f}(\psi)(y) = (f \nearrow \psi)(y).$$

Put $\Pi^g(f) = f \odot g$ for a fixed $g \in L_2^Y$. By Lemma 4.1 (1), $\Pi^g(\bigvee_{i\in\Gamma} f_i) = (\bigvee_{i\in\Gamma} f_i) \odot g = \bigvee_{i\in\Gamma}(f_i \odot g) = \bigvee_{i\in\Gamma} \Pi^g(f_i)$. Define $\mathcal{H}_{\Pi^g} : (L_3^Y)^X \to L_1^X$ as

$$\mathcal{H}_{\Pi^g}(\psi) = \bigvee\{f \in L_1^X \mid \Pi^g(f)(x)(y) \leq_3 \psi(x)(y)\}.$$

Since $\bigvee_{x\in X}(f(x) \odot g(y)) \leq_3 \psi(x)(y)$, $f(x) \leq_1 g(y) \searrow \psi(x)(y)$. Hence $\mathcal{H}_{\Pi^g}(\psi)(x) \leq_1 \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y))$.

On the other hand, since $\Pi^g(\bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)))(x) = (\bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y))) \odot g(y)) \leq_3 (g(y) \searrow \psi(x)(y)) \odot g(y) \leq_3 \psi(x)(y)$, we have $\mathcal{H}_{\Pi^g}(\psi)(x) \geq_1 \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y))$. Thus, $\mathcal{H}_{\Pi^g}(\psi)(x) = \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y))$. It follows

$$\Pi^g(f)(x)(y) = f(x) \odot g(y) \leq_3 \psi(x)(y)$$
$$\text{iff } f(x) \leq_1 \mathcal{H}_{\Pi^g}(\psi)(x) = (g \nearrow \psi)(x).$$

□

Remark 4.3. Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$. Let X be a set of objects, Y be a set of attributes and the triple $(X, Y, \psi \in (L_3^Y)^X)$ be a fuzzy information system. From the adjoint triple $(\odot, \nearrow, \searrow)$ with respect to $(L_1^X, \leq_1)$, $(L_2^Y, \leq_2)$ and $((L_3^Y)^X, \leq_3)$ in above theorem, we construct the concept-forming operators (ref.[6,13,14]) $F : L_1^X \to L_2^Y$ and $G : L_2^Y \to L_1^X$ are defined as follows

$$F(f)(y) = (f \nearrow \psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)),$$
$$G(g)(x) = (g \searrow \psi)(x) = \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)).$$

Theorem 4.4. *Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$. Define $\odot : L_1^X \times (L_2^Y)^X \to L_3^Y$ as $(f \odot \phi)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y))$. Then there exist $\nearrow: L_1^X \times L_3^Y \to (L_2^Y)^X$ and $\searrow: (L_2^Y)^X \times L_3^Y \to L_1^X$ defined as*

$$(f \nearrow h)(x)(y) = f(x) \nearrow h(y),$$
$$(\phi \searrow h)(x) = \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)).$$

Moreover, $(\odot, \nearrow, \searrow)$ is an adjoint triple with respect to $(L_1^X, \leq_1)$, $((L_2^Y)^X, \leq_2)$ and $(L_3^Y, \leq_3)$ such that

$$f \leq_1 \phi \searrow h \text{ iff } f \odot \phi \leq_3 h \text{ iff } \phi \leq_2 f \nearrow h.$$

Proof. Put $\Delta_f(\phi) = f \odot \phi$ for a fixed $f \in L_1^X$. and $\Delta^\phi(f) = f \odot \phi$. By Lemma 4.1 (1), $\Delta_f(\bigvee_{i\in\Gamma} \phi_i) = f \odot (\bigvee_{i\in\Gamma} \phi_i) = \bigvee_{i\in\Gamma}(f \odot \phi_i) = \bigvee_{i\in\Gamma} \Delta_f(\phi_i)$. Define $\mathcal{J}_{\Delta_f} : L_3^Y \to (L_2^Y)^X$ as

$$\mathcal{J}_{\Delta_f}(h) = \bigvee\{\phi \in (L_2^Y)^X \mid \Delta_f(\phi)(y) \leq_3 h(y)\}.$$

Since $\Delta_f(\phi)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y)) \leq_3 h(y)$, $\phi(x)(y) \leq_2 f(x) \nearrow h(y)$. Hence $\mathcal{J}_{\Delta_f}(h)(x)(y) \leq_2 f(x) \nearrow h(y)$.

For $(f \nearrow h) \in (L_2^Y)^X$ such that $(f \nearrow h)(x)(y) = f(x) \nearrow h(y)$, since $\Delta_f(f \nearrow h)(y) = \bigvee_{x\in X}(f(x) \odot (f(x) \nearrow h(y))) \leq_3 h(y)$, we have $\mathcal{J}_{\Delta_f}(h)(x)(y) \geq_2 f(x) \nearrow h(y)$. Thus, $\mathcal{J}_{\Delta_f}(h)(x)(y) = f(x) \nearrow h(y)$. It follows

$$\Delta_f(\phi)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y)) \leq_3 h(y)$$
$$\text{iff } \phi(x)(y) \leq_2 \mathcal{J}_{\Delta_f}(h)(x)(y) = f(x) \nearrow h(y).$$

Put $\Delta^\phi(f) = f \odot \phi$ a fixed $\phi \in (L_2^Y)^X$. By Lemma 4.1 (1), $\Delta^\phi(\bigvee_{i\in\Gamma} f_i) = (\bigvee_{i\in\Gamma} f_i) \odot \phi = \bigvee_{i\in\Gamma}(f_i \odot \phi) = \bigvee_{i\in\Gamma} \Delta^\phi(f_i)$. Define $\mathcal{J}_{\Delta^\phi} : L_3^Y \to L_1^X$ as

$$\mathcal{J}_{\Delta^\phi}(h) = \bigvee\{f \in L_1^X \mid \Delta^\phi(f)(y) \leq_3 h(y)\}.$$

Since $\Delta^{\phi}(f)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y)) \leq_3 h(y)$, $f(x) \leq_1 \phi(x)(y) \searrow h(y)$. Hence $\mathcal{J}_{\Delta^{\phi}}(h)(x) \leq_1 \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y))$.

On the other hand, since $\Delta^{\phi}(\bigwedge_{z\in Y}(\phi(-)(z) \searrow h(z))(y) = \bigvee_{x\in X}(\bigwedge_{z\in Y}(\phi(x)(z) \searrow h(z))\odot\phi(x)(y)) \leq_3 \bigvee_{x\in X}((\phi(x)(y) \searrow h(y))\odot\phi(x)(y)) \leq_3 h(y)$, we have $\mathcal{J}_{\Delta^{\phi}}(h)(x) \geq_1 \bigvee_{y\in Y}(\phi(x)(y) \searrow h(y))$. Thus, $\mathcal{J}_{\Delta^{\phi}}(h)(x) = \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y))$. It follows

$$\begin{array}{l}\Delta^{\phi}(f)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y)) \leq_3 h(y)\\ \text{iff } f(x) \leq_2 \mathcal{J}_{\Delta^{\phi}}(h)(x) = \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)).\end{array}$$

□

REMARK 4.5. Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$. Let X be a set of objects, Y be a set of attributes and the triple $(X, Y, \phi \in (L_2^Y)^X)$ be a fuzzy information system. From the adjoint triple $(\odot, \nearrow, \searrow)$ with respect to $(L_1^X, \leq_1)$, $((L_2^Y)^X, \leq_2)$ and $(L_3^Y, \leq_3)$ in above theorem, we construct a fuzzy relational erosion and fuzzy relational dilation (ref.[6,13,14]) with respect to ϕ, $\epsilon_\phi : L_3^Y \to L_1^X$ and $\delta_\phi : L_1^X \to L_3^Y$ is defined as

$$\begin{array}{l}\epsilon_\phi(h)(x) = (\phi \searrow h)(x) = \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)),\\ \delta_\phi(f)(y) = (f \odot \phi)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y)).\end{array}$$

From the above theorem, we similarly prove the following corollary.

COROLLARY 4.6. *Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$. Define $\odot : (L_1^Y)^X \times L_2^Y \to L_3^X$ as $(\theta \odot g)(x) = \bigvee_{x\in X}(\theta(x)(y) \odot g(y))$. Then there exist $\nearrow: (L_1^Y)^X \times L_3^X \to L_2^Y$ and $\searrow: L_2^Y \times L_3^X \to (L_1^Y)^X$ defined as*

$$\begin{array}{l}(\theta \nearrow h)(x)(y) = \bigwedge_{x\in X}(\theta(x)(y) \nearrow h(x)),\\ (g \searrow h)(x)(y) = g(y) \searrow h(x).\end{array}$$

Moreover, $(\odot, \nearrow, \searrow)$ is an adjoint triple with respect to $((L_1^Y)^X, \leq_1)$, $(L_2^Y, \leq_2)$ and $(L_3^X, \leq_3)$ such that

$$\theta \leq_1 g \searrow h \text{ iff } \theta \odot g \leq_3 h \text{ iff } g \leq_2 \theta \nearrow h.$$

REMARK 4.7. Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. Let X be a set of objects, Y be a set of attributes and the triple $(X, Y, \theta \in (L_1^Y)^X)$ be a fuzzy information system. From the adjoint triple $(\odot, \nearrow, \searrow)$ with respect to $((L_1^Y)^X, \leq_1)$, $(L_2^Y, \leq_2)$ and $(L_3^X, \leq_3)$ in above corollary, we construct a fuzzy relational property-oriented erosion and fuzzy relational property-oriented dilation (ref.[6,13,14]) with respect to θ, $\epsilon_\theta : L_3^X \to L_2^Y$ and $\delta_\phi : L_2^Y \to L_3^X$ is defined as

$$\begin{aligned}\epsilon_\theta(h)(y) &= (\theta \nearrow h)(y) = \bigwedge_{x\in X}(\theta(x)(y) \nearrow h(x)),\\ \delta_\theta(f)(x) &= (\theta \odot g)(x) = \bigvee_{y\in Y}(\theta(x)(y) \odot g(y)).\end{aligned}$$

THEOREM 4.8. *Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$ and X, Y be sets. Define $\nearrow$: $L_1^X \times (L_3^Y)^X \to L_2^Y$ as*

$$(f \nearrow \psi)(y) = \bigwedge_{x\in X} (f(x) \nearrow \psi(x)(y)).$$

Then there exist $\searrow$: $L_2^Y \times (L_3^Y)^X \to L_1^X$ and $\odot : L_1^X \times L_2^Y \to (L_3^Y)^X$ defined as

$$\begin{aligned}(g \searrow \psi)(x) &= \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)).\\ (f \odot g)(x)(y) &= f(x) \odot g(y).\end{aligned}$$

Moreover, $(\odot, \nearrow, \searrow)$ is an adjoint triple with respect to $(L_1^X, \leq_1)$, $(L_2^Y, \leq_2)$ and $((L_3^Y)^X, \leq_3)$ such that

$$f \leq_1 g \searrow \psi \ \text{ iff } f \odot g \leq_3 \psi \ \text{ iff } g \leq_2 f \nearrow \psi.$$

Proof. Put $\Theta^\psi(f)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y))$ for a fixed map $\psi \in (L_3^Y)^X$. By Lemma 4.1(2), $\Theta^\psi(\bigvee_{i\in\Gamma} f_i)(y) = \bigwedge_{x\in X}((\bigvee_{i\in\Gamma} f_i(x)) \nearrow \psi(x)(y)) = \bigwedge_{x\in X}(\bigwedge_{i\in\Gamma}(f_i(x) \nearrow \psi(x)(y))) = \bigwedge_{i\in\Gamma} \Theta^\psi(f_i)(y)$. Define $\mathcal{K}_{\Theta^\psi} : L_2^Y \to L_1^X$ as

$$\mathcal{K}_{\Theta^\psi}(g) = \bigvee\{f \in L_1^X \mid \Theta^\psi(f)(y) \geq_2 g(y)\}.$$

Since $\Theta^\psi(f)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)) \geq_2 g(y)$, $g(y) \leq_2 f(x) \nearrow \psi(x)(y)$ implies $f(x) \odot g(y) \leq_3 \psi(x)(y)$. Thus $f(x) \leq_1 \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)) = (g \searrow \psi)(x)$. Hence $\mathcal{K}_{\Theta^\psi}(g)(x) \leq_1 \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)) = (g \searrow \psi)(x)$.

On the other hand, since $\Theta^\psi(g \searrow \psi)(y) = \bigwedge_{z\in X}(\bigwedge_{y\in Y}(g(y) \searrow \psi(z)(y)) \nearrow \psi(z)(y)) \geq_2 \bigwedge_{z\in X}((g(y) \searrow \psi(z)(y)) \nearrow \psi(z)(y)) \geq_2 g(y)$, we have $\mathcal{K}_{\Theta^\psi}(g)(x) \geq_1 \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)) = (g \searrow \psi)(x)$. Thus, $\mathcal{K}_{\Theta^\psi}(g)(x) = \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)) = (g \searrow \psi)(x)$. It follows

$$\begin{aligned}&(f \nearrow \psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)) = \Theta^\psi(f)(y) \geq_2 g(y)\\ &\text{iff } \mathcal{K}_{\Theta^\psi}(g)(x) = (g \searrow \psi)(x) \geq f(x).\end{aligned}$$

Put $\Theta_f(\psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y))$ for a fixed map $f \in L_1^X$. By Lemma 4.1 (2), $\Theta_f(\bigwedge_{i\in\Gamma} \psi_i) = \bigwedge_{x\in X}(f(x) \nearrow \bigwedge_{i\in\Gamma} \psi_i(x)(y)) =$

$\bigwedge_{i\in\Gamma}\bigwedge_{x\in X}(f(x) \nearrow \psi_i(x)(y)) = \bigwedge_{i\in\Gamma}\Theta_f(\psi_i)$. Define $\mathcal{K}_{\Theta_f} : L_2^Y \to (L_3^Y)^X$ as

$$\mathcal{K}_{\Theta_f}(g) = \bigvee\{\psi \in (L_3^Y)^X \mid \Theta_f(\psi)(y) \geq_2 g(y)\}.$$

Since $\Theta_f(\psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)) \geq_2 g(y)$, $g(y) \leq_2 f(x) \nearrow \psi(x)(y)$ implies $f(x) \odot g(y) \leq_3 \psi(x)(y)$. Thus $\mathcal{K}_{\Theta_f}(g)(x)(y) \geq_3 f(x) \odot g(y)$.

On the other hand, since $\Theta_f(f \odot g)(y) = \bigwedge_{x\in X}(f(x) \nearrow (f(x) \odot g(y))) \geq_2 g(y)$, we have $\mathcal{K}_{\Theta_f}(g)(x)(y) \geq_3 f(x)\odot g(y)$. Thus, $\mathcal{K}_{\Theta_f}(g)(x)(y) = f(x) \odot g(y)$. It follows

$$\begin{array}{l}(f \nearrow \psi)(y) = \bigwedge_{x\in X}(f(x) \nearrow \psi(x)(y)) = \Theta_f(\psi)(y) \geq_2 g(y) \\ \text{iff } \mathcal{K}_{\Theta_f}(g)(x)(y) = f(x) \odot g(y) \geq_3 \psi(x)(y).\end{array}$$

□

THEOREM 4.9. *Let $(\odot, \nearrow, \searrow)$ be an adjoint triple with respect to $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(L_3, \leq_3)$. Define $\nearrow: L_1^X \times L_3^Y \to (L_2^Y)^X$ as*

$$(f \nearrow h)(x)(y) = f(x) \nearrow h(y).$$

Then there exist $\searrow: (L_2^Y)^X \times L_3^Y \to L_1^X$ and $\odot : L_1^X \times (L_2^Y)^X \to L_3^Y$ defined as

$$\begin{array}{l}(\phi \searrow h)(x) = \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)), \\ (f \odot \phi)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y)).\end{array}$$

Moreover, $(\odot, \nearrow, \searrow)$ is an adjoint triple with respect to $(L_1^X, \leq_1)$, $((L_2^Y)^X, \leq_2)$ and $(L_3^Y, \leq_3)$ such that

$$f \leq_1 \phi \searrow h \text{ iff } f \odot \phi \leq_3 h \text{ iff } \phi \leq_2 f \nearrow h.$$

Proof. Put $\Phi^h(f)(x)(y) = f(x) \nearrow h(y)$ for a fixed map $h \in L_3^Y$. By Lemma 4.1 (2), $\Phi^h(\bigvee_{i\in\Gamma} f_i)(x)(y) = (\bigvee_{i\in\Gamma} f_i(x)) \nearrow h(y) = \bigwedge_{i\in\Gamma}(f_i(x) \nearrow h(y)) = \bigwedge_{i\in\Gamma}\Phi^h(f_i)(x)(y)$. Define $\mathcal{P}_{\Phi^h} : (L_2^Y)^X \to L_1^X$ as

$$\mathcal{P}_{\Phi^h}(\phi) = \bigvee\{f \in L_1^X \mid \Phi^h(f)(x)(y) \geq_2 \phi(x)(y)\}.$$

Since $\Phi^h(f)(x)(y) = f(x) \nearrow h(y) \geq_2 \phi(x)(y)$, $f(x) \odot \phi(x)(y) \leq_3 h(y)$. Thus $f(x) \leq_1 \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)) = (\phi \searrow h)(x)$. Hence $\mathcal{P}_{\Phi^h}(\phi)(x) \leq_1 \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)) = (\phi \searrow h)(x)$.

Since $\Phi^h(\phi \searrow h)(x)(y) = \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)) \nearrow h(y) \geq_2 (\phi(x)(y) \searrow h(y)) \nearrow h(y) \geq_2 \phi(x)(y)$, we have $\mathcal{P}_{\Phi^h}(\phi)(x) \geq_1 \bigwedge_{y\in Y}(\phi(x)(y) \searrow$

$h(y)) = (\phi \searrow h)(x)$. Thus, $\mathcal{P}_{\Phi^h}(\phi)(x) = \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)) = (\phi \searrow h)(x)$. It follows

$$\begin{array}{l}(f \nearrow h)(x)(y) = f(x) \nearrow h(y) = \Phi^h(f)(x)(y) \geq_2 \phi(x)(y)\\ \text{iff } \mathcal{P}_{\Phi^h}(\phi)(x) = \bigwedge_{y\in Y}(\phi(x)(y) \searrow h(y)) = (\phi \searrow h)(x) \geq_1 f(x).\end{array}$$

Put $\Phi_f(h)(x)(y) = f(x) \nearrow h(y)$ for a fixed map $f \in L_1^X$. By Lemma 4.1 (2), $\Phi_f(\bigwedge_{i\in\Gamma} h_i)(x)(y) = f(x) \nearrow \bigwedge_{i\in\Gamma} h_i(y) = \bigwedge_{i\in\Gamma}(f(x) \nearrow h_i(y)) = \bigwedge_{i\in\Gamma} \Phi_f(h_i)(x)(y)$. Define $\mathcal{P}_{\Phi_f} : (L_2^Y)^X \to L_1^X$ as

$$\mathcal{P}_{\Phi_f}(\phi) = \bigwedge\{h \in L_3^Y \mid \Phi_f(h)(x)(y) \geq_2 \phi(x)(y)\}.$$

Since $\Phi_f(h)(x)(y) = f(x) \nearrow h(y) \geq_2 \phi(x)(y)$, $f(x) \odot \phi(x)(y) \leq_2 h(y)$. Thus $\mathcal{P}_{\Phi_f}(\phi) \geq_3 \bigvee_{x\in X}(f(x) \odot \phi(x)(y))$.

Since $\Phi_f(\bigvee_{x\in X}(f(x) \odot \phi(x)(-)))(x)(y) = f(x) \nearrow \bigvee_{x\in X}(f(x) \odot \phi(x)(y)) \geq_2 f(x) \nearrow (f(x) \odot \phi(x)(y)) \geq_2 \phi(x)(y)$, we have $\mathcal{P}_{\Phi_f}(\phi)(y) \leq_3 \bigvee_{x\in X}(f(x) \odot \phi(x)(y))$. Thus, $\mathcal{P}_{\Phi_f}(\phi)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y))$. It follows

$$\begin{array}{l}(f \nearrow h)(x)(y) = f(x) \nearrow h(y) = \Phi_f(h)(x)(y) \geq_2 \phi(x)(y)\\ \text{iff } \mathcal{P}_{\Phi_f}(\phi)(y) = \bigvee_{x\in X}(f(x) \odot \phi(x)(y)) = (f \odot \phi)(y) \leq_3 h(y).\end{array}$$

□

From the above theorems, we similarly prove the following corollaries.

COROLLARY 4.10. *Let* $(\odot, \nearrow, \searrow)$ *be an adjoint triple with respect to* $(L_1, \leq_1)$, $(L_2, \leq_2)$ *and* $(L_3, \leq_3)$ *and* X, Y *be sets. Define* $\searrow: L_2^Y \times (L_3^Y)^X \to L_1^X$ *as*

$$(g \searrow \psi)(x) = \bigwedge_{y\in Y}(g(y) \searrow \psi(x)(y)).$$

Then there exist $\nearrow: L_1^X \times (L_3^Y)^X \to L_2^Y$ *and* $\odot : L_1^X \times L_2^Y \to (L_3^Y)^X$ *defined as*

$$\begin{array}{l}(f \nearrow \psi)(y) = \bigwedge_{x\in X}(f(x) \searrow \psi(x)(y)).\\ (f \odot g)(x)(y) = f(x) \odot g(y).\end{array}$$

Moreover, $(\odot, \nearrow, \searrow)$ *is an adjoint triple with respect to* $(L_1^X, \leq_1)$, $(L_2^Y, \leq_2)$ *and* $((L_3^Y)^X, \leq_3)$ *such that*

$$f \leq_1 g \searrow \psi \ \text{ iff } f \odot g \leq_3 \psi \ \text{ iff } g \leq_2 f \nearrow \psi.$$

Corollary 4.11. *Let* $(\odot, \nearrow, \searrow)$ *be an adjoint triple with respect to* $(L_1, \leq_1)$, $(L_2, \leq_2)$ *and* $(L_3, \leq_3)$. *Define* $\searrow\colon (L_2^Y)^X \times L_3^Y \to L_1^X$ *as*

$$(\phi \searrow h)(x) = \bigwedge_{y \in Y} (\phi(x)(y) \searrow h(y)).$$

Then there exist $\nearrow\colon L_1^X \times L_3^Y \to (L_2^Y)^X$ *and* $\odot : L_1^X \times (L_2^Y)^X \to L_3^Y$ *defined as*

$$(f \nearrow h)(x)(y) = f(x) \nearrow h(y),$$
$$(f \odot \phi)(y) = \bigvee_{x \in X} (f(x) \odot \phi(x)(y)).$$

Moreover, $(\odot, \nearrow, \searrow)$ *is an adjoint triple with respect to* $(L_1^X, \leq_1)$, $((L_2^Y)^X, \leq_2$ $)$ *and* $(L_3^Y, \leq_3)$ *such that*

$$f \leq_1 \phi \searrow h \text{ iff } f \odot \phi \leq_3 h \text{ iff } \phi \leq_2 f \nearrow h.$$

Corollary 4.12. *Let* $(\odot, \nearrow, \searrow)$ *be an adjoint triple with respect to* $(L_1, \leq_1)$, $(L_2, \leq_2)$ *and* $(L_3, \leq_3)$. *Define* $\nearrow\colon (L_1^Y)^X \times L_3^X \to L_2^Y$ *as*

$$(\theta \nearrow h)(x)(y) = \bigwedge_{x \in X} (\theta(x)(y) \nearrow h(x)).$$

Then there exist $\odot : (L_1^Y)^X \times L_2^Y \to L_3^X$ *and* $\searrow\colon L_2^Y \times L_3^X \to (L_1^Y)^X$ *defined as*

$$(\theta \odot g)(x) = \bigvee_{x \in X} (\theta(x)(y) \odot g(y)),$$
$$(g \searrow h)(x)(y) = g(y) \searrow h(x).$$

Moreover, $(\odot, \nearrow, \searrow)$ *is an adjoint triple with respect to* $((L_1^Y)^X, \leq_1)$, $(L_2^Y, \leq_2)$ *and* $(L_3^X, \leq_3)$ *such that*

$$\theta \leq_1 g \searrow h \text{ iff } \theta \odot g \leq_3 h \text{ iff } g \leq_2 \theta \nearrow h.$$

Corollary 4.13. *Let* $(\odot, \nearrow, \searrow)$ *be an adjoint triple with respect to* $(L_1, \leq_1)$, $(L_2, \leq_2)$, $(L_3, \leq_3)$. *Define* $\searrow\colon L_2^Y \times L_3^X \to (L_1^Y)^X$ *as*

$$(g \searrow h)(x)(y) = g(y) \searrow h(x).$$

Then there exist $\odot : (L_1^Y)^X \times L_2^Y \to L_3^X$ *and* $\nearrow\colon (L_1^Y)^X \times L_3^X \to L_2^Y$ *defined as*

$$(\theta \odot g)(x) = \bigvee_{x \in X} (\theta(x)(y) \odot g(y)),$$
$$(\theta \nearrow h)(x)(y) = \bigwedge_{x \in X} (\theta(x)(y) \nearrow h(x)).$$

Moreover, $(\odot, \nearrow, \searrow)$ *is an adjoint triple with respect to* $((L_1^Y)^X, \leq_1)$, $(L_2^Y, \leq_2)$ *and* $(L_3^X, \leq_3)$ *such that*

$$\theta \leq_1 g \searrow h \text{ iff } \theta \odot g \leq_3 h \text{ iff } g \leq_2 \theta \nearrow h.$$

EXAMPLE 4.14. Let $X = \{x, y\}$ be a set of cars and $Y = \{a, b, c\}$ be a set of attributes. Let $(L_1 = \{0, \frac{1}{2}, 1\}, 0 <_1 \frac{1}{2} <_1 1)$, $(L_2 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}, 0 <_2 \frac{1}{3} <_2 \frac{2}{3} <_2 1)$ and $(L_3 = \{0, 1, 2\}, 0 <_3 1 <_3 2)$ be lattices. Define $\odot : L_1 \times L_2 \to L_3$ as follows:

$\odot$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	0	0	1	2
$\frac{1}{2}$	0	1	1	2
1	0	1	2	2

We can define $p \nearrow r = \bigvee\{q \in L_2 \mid p \odot q \leq_3 r\}$ and $q \searrow r = \bigvee\{p \in L_1 \mid p \odot q \leq_3 r\}$ as follows

$\searrow$	0	1	2
0	1	1	1
$\frac{1}{3}$	0	1	1
$\frac{2}{3}$	0	$\frac{1}{2}$	1
1	0	0	1

$\nearrow$	0	1	2
0	1	1	1
$\frac{1}{2}$	0	$\frac{2}{3}$	1
1	0	$\frac{1}{3}$	1

Then $(\odot, \nearrow, \searrow)$ is an adjoint triple.

(1) For $f = (f(x), f(y)) = (1, \frac{1}{2}) \in L_1^X$, $g = (g(a), g(b), g(c)) = (\frac{1}{3}, 0, \frac{2}{3}) \in L_2^Y$ and $\psi(x) : Y \to L_3$ for $x \in \{x, y\}$ with $\psi(x)(a) = \psi(x, a)$ as

$$f \odot g = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} \leq_3 \psi = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

From Theorem 4.2, we have

$$f = (1, \tfrac{1}{2}) \leq_1 (g \searrow \psi)(-) = \textstyle\bigwedge_{y \in Y}(g(y) \searrow \psi(-)(y)) = (1, 1),$$
$$g = (\tfrac{1}{3}, 0, \tfrac{2}{3}) \leq_2 (f \nearrow \psi)(-) = \textstyle\bigwedge_{x \in X}(f(x) \searrow \psi(x)(-)) = (1, 0, \tfrac{2}{3}).$$

(2) For $f = (f(x), f(y)) = (1, \frac{1}{2}) \in L_1^X$, $h = (h(a), h(b), h(c)) = (0, 2, 1) \in L_3^Y$ and $\phi(x) : Y \to L_3$ for $x \in \{x, y\}$ with $\psi(x)(a) = \psi(x, a)$ as

$$\phi = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \leq_2 (f \searrow h) = \begin{pmatrix} 0 & 1 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \end{pmatrix}.$$

From Theorem 4.3, we have

$$f = (1, \tfrac{1}{2}) \leq_1 (\phi \searrow h)(-) = \textstyle\bigwedge_{y \in Y}(\phi(-)(y) \searrow h(y)) = (1, 1),$$
$$(f \odot \psi)(-) = \textstyle\bigvee_{x \in X}(f(x) \odot \psi(x)(-)) = (0, 2, 1) \leq_3 h = (0, 2, 1).$$

(3) For $g = (\frac{1}{3}, 0, \frac{2}{3}) \in L_2^Y$, $h = (h(x), h(y)) = (1, 2) \in L_3^X$ and $\phi(x) : Y \to L_3$ for $x \in \{x, y\}$ with $\theta(x)(a) = \theta(x, a)$ as

$$\Theta = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 1 \end{pmatrix} \leq_2 (g \nearrow h) = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}$$

From Corollary 4.4, we have

$$g = (\tfrac{1}{3}, 0, \tfrac{2}{3}) \leq_2 (\theta \nearrow h)(-) = \bigwedge_{x\in X}(\theta(x)(-) \nearrow h(x)) = (1, \tfrac{1}{3}, \tfrac{2}{3}),$$
$$(\theta \odot g)(-) = \bigvee_{y\in Y}(\psi(-)(y) \odot g(y)) = (1,2) \leq_1 h = (1,2).$$

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